Aggregation of Preferences

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In several past decades various abstract models have been created for the purpose of assisting individuals or group of individuals in arriving at good decisions. Here we are concerned with some questions that arise when a group of individuals has to select one alternative from a given set of possible alternatives. Depending on the context the alternatives might be called strategies, commodity bundles, feasible solutions and so forth.

If each member of the group has logically consistent preferences and if the preferences of all members are sufficiently similar, then there is usually easy to find a satisfactory solution by using a suitable voting procedure. Unfortunately, in many cases the latter condition is not satisfied. Then one of the possibilities for resolving conflict among preferences of different members is to employ a suitable rule for aggregating the individual preferences into a group preference, and then to select the best alternative on the basis of the resulting group preference. To illustrate some obstacles and difficulties we may encounter, we first consider the following simple example.

Let $X = \{x, y, z\}$ be a set consisting of three different alternatives x, y and z, and let $N = \{1, 2, 3\}$ be a group consisting of three members, which we will call players. Suppose that

player 1 prefers x to y, y to z, and hence x to z,
player 2 prefers z to x, x to y, and hence z to y,
player 3 prefers y to z, z to x, and hence y to x.

We wish to aggregate this profile of three individual preference relations over set X into a group preference relation over the same set X in such a way that the individual preferences are taken into account.

Let us apply one of the common aggregation rules, namely, the simple majority rule. In our particular situation we obtain

x is better than y because 1 and 2 vote for x over y,
y is better than z because 1 and 3 vote for y over z,
z is better than x because 2 and 3 vote for z over x.

We now have rather unpleasant situation. Each individual preference relation is without any contradiction however the resulting group preference relation is self-contradictory. It is cyclic and no alternative is best. In other words, all participants are rational, however as a group they are rather irrational. This phenomenon is known as the Condorcet voting paradox.

We should not conclude from the example that the simple majority rule is not applicable at all. For instance, if

player 1 prefers x to y, y to z, and x to z,
player 2 prefers y to x, x to z, and y to z,
player 3 prefers x to z, z to y, and x to y,

then we have the situation in which the simple majority rule produces perfectly acceptable result: namely,

x is better than y, y is better than z, x is better than z.

Before proceeding to more detailed analysis, we need to introduce some definitions and notation. Let X denote the set whose elements are to be evaluated in terms of preferences in some particular situation. We shall call the elements of X alternatives. If a player prefers an alternative x to alternative y, we shall write briefly $x \succ y$. We assume that each player is rational in the sense that his or her preferences satisfy certain conditions which reflect some reasonable requirements of consistency. For example, it would be rather strange to allow the existence of two alternatives, say x and y, such that a player prefers alternative x to alternative y and at the same time he or she prefers alternative y to alternative x. In other words, we shall assume that preference relations of all players are asymmetric in the sense that, for all $x, y \in X$, we have

$$\text{if } x \succ y, \text{ then not } x \succ y. \qquad \text{(asymmetry)}$$

As a consequence, we obtain that preference relations of all players are irreflexive in the sense that, for all $x \in X$, we have

$$\text{not } x \succ x. \qquad \text{(irreflexivity)}$$

It would also be rather unreasonable to allow that there are three alternatives, say x, y, and z, such that a player prefers x to y, y to z, and simultaneously z to x. Therefore we assume that the preference relations are transitive, that is,

$$\text{if } x \succ y \text{ and } y \succ z, \text{ then } x \succ z. \qquad \text{(transitivity)}$$

We also need the property of negative transitivity, which means that

$$\text{if not } x \succ y \text{ and not } y \succ z, \text{ then not } x \succ z. \quad \text{(neg. transitivity)}$$

Let us now return to the problem of aggregation of individual preferences into a group preference. We have already demonstrated that the simple majority rule may aggregate transitive preference relations into an intransitive relation, even in a very simple situation involving only three alternatives and three players. In this connection, two natural questions arise. First, whether such an undesirable property is typical for the simple majority rule, and how to avoid it. Second, whether there exist some reasonable rules without that drawback.

The following table gives a partial answer to the first question. Its entries present the fractions of the set of all n-tuples of orders on the set of m alternatives for which the simple majority rule gives an intransitive relation.

m \ n	3	5	7	9	11	⋯	limit
3	.056	.069	.075	.078	.080	…	.088
4	.111	.139	.150	.156	.160	…	.176
5	.160	.200	.215	.230	.251	…	.251
6	.202	.255	.258	.284	.294	…	.315
7	.239	.299	.305	.342	.343	…	.369
⋮	⋮	⋮	⋮	⋮	⋮		⋮
limit	1.000	1.000	1.000	1.000	1.000	…	1.000

We see that that the probability of encountering the paradox increases rather rapidly with growing number of alternatives. One way of avoiding the paradox is to limit the freedom of players to choose arbitrary preferences.

Then it is desirable to restrict the players freedom as less as possible. This leads to an interesting problem, whose many aspects are still waiting for a satisfactory solution. To facilitate its formulation, let us assume that the players are never indifferent between distinct alternatives, so that their preferences are represented by permutations of alternatives. In the simplest nontrivial case of three alternatives x, y and z, each player's preference relation is represented by one of the orders

$$xyz, xzy, yxz, yzx, zxy, zyx.$$

Let us forbid one of the orders, say xyz. Does it help? Not really, as we can see from the following preferences. Suppose that

player 1 prefers y to x, x to z, and y to z,
player 2 prefers x to z, z to y, and x to y,
player 3 prefers z to y, y to x, and z to x.

Then in the simple majority voting y wins over x, x wins over z and z wins over y. Since similar results hold also for other orders, we have to forbid more than one order to avoid the paradox. It is clear from the previous examples that we are not completely free in excluding orders and that it is necessary to exclude one of the orders xyz, zxy, yzx and one of the orders zyx, xzy, yxz. It turns out that this necessary condition is also sufficient.

Thus, for example, the set consisting of the orders xyz, zxy, xzy, yxz has the property that (independently of how many players we have) if the preference relation of each player belongs to this set, then the group preference resulting from the simple majority voting is acyclic. Let us call such sets of orders acyclic sets. It is surprisingly difficult to specify the largest (in the sense of the number of orders) acyclic sets when the number of alternatives is greater than three. It is known that the largest acyclic sets for four alternatives consists of nine orders, and the largest acyclic sets for five alternatives consists of twenty orders. But it is not much known about the exact size of the largest acyclic sets when the number of alternatives is greater than five. The reader desiring to enjoy research in this area can find details in the paper by Peter C. Fishburn[1].

To answer the second question, we need to define our problem more precisely. Namely, we have to specify which aggregation rules are considered as reasonable.

Let X be a given set of alternatives and let N be the set of individuals in society. We assume that both X and N are finite sets containing at least two elements. For definiteness, let N be the set $\{1, 2, \ldots, n\}$. The set of all negatively transitive asymmetric relations on X will be denoted by P, and ordered n-tuples $(\succ_1, \succ_2, \cdots, \succ_n)$ of relations from P will be called profiles. We assume that each person is completely free to have an arbitrary relation from P as his or her preference relation $\succ_i$. We are interested in functions F that map the set of all possible profiles, that is, the Cartesian product $P^n=(P\times P\times\cdots\times P)$, into the set P. Such functions are called *social welfare functions*, and one is interested in those functions that have some reasonable properties. As examples of quite reasonable properties, consider the following three conditions on the behavior of social welfare functions.

CONDITION 1. *If every member of a group prefers alternative x to alternative y, then the group also prefers x to y.*

Formally speaking, we require that for all alternatives $x, y \in X$ and all profiles $p=(\succ_1, \succ_2, \cdots, \succ_n)$ in P^n, we have:

if $x \succ_i y$ for all $i \in N$, then $xF(p)y$.

CONDITION 2. *There is no dictator.*

A person $i \in N$ is called a dictator when, for all alternatives $x, y \in X$ and all profiles $p=(\succ_1, \succ_2, \cdots, \succ_n)$ from P^n, we have:

if $x \succ_i y$, then $xF(p)y$.

It is easy to see that the second condition can be expressed as follows: For each person $i \in N$, there exist alternatives x and y in X, and a profile $p=(\succ_1, \succ_2, \cdots, \succ_n)$ in P^n such that

$x \succ_i y$ and not $xF(p)y$.

CONDITION 3. *The group preference between any two alternatives depends only on the individual preferences between those two alternatives.*

In order to formalize this condition, we first introduce some notation. If Y is a subset of X, then it is possible that two profiles, say $p=(\succ_1,\succ_2,\cdots,\succ_n)$ and $p'=(\succ'_1,\succ'_2,\cdots,\succ'_n)$ are different when considered on X, but at the same time they may be identical when considered on Y only. Let p/Y and $F(p)/Y$ denote the reductions of p and $F(p)$ from X to Y, respectively. Using similar notation for other profiles and relations, we can now express the third condition as follows:

For all $x, y \in X$ and all $p, p' \in P^n$,

if $p/\{x,y\}=p'/\{x,y\}$, then $F(p)/\{x,y\}=F(p')/\{x,y\}$

It turns out that in spite of the fact that each condition is very appealing on its own, they cannot be satisfied simultaneously.

THEOREM (Arrow). *If X has more than two elements, then there is no $F:P^n \rightarrow P$ satisfying all three conditions.*

There is an extensive literature on the problem of aggregation of individual preferences into a group or social preference. For information about various aspects and recent status of research, see the following handbook edited by K.J. Arrow, A.K. Sen and K. Suzumura[2].

1 "Decision theory and discrete mathematics", Discrete Applied Mathematics 68 (1996) 209-221.

2 "Handbook of Social Choice and Welfare", Elsevier, Amsterdam, Volume 1, 2002; Volume 2, forthcoming.

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